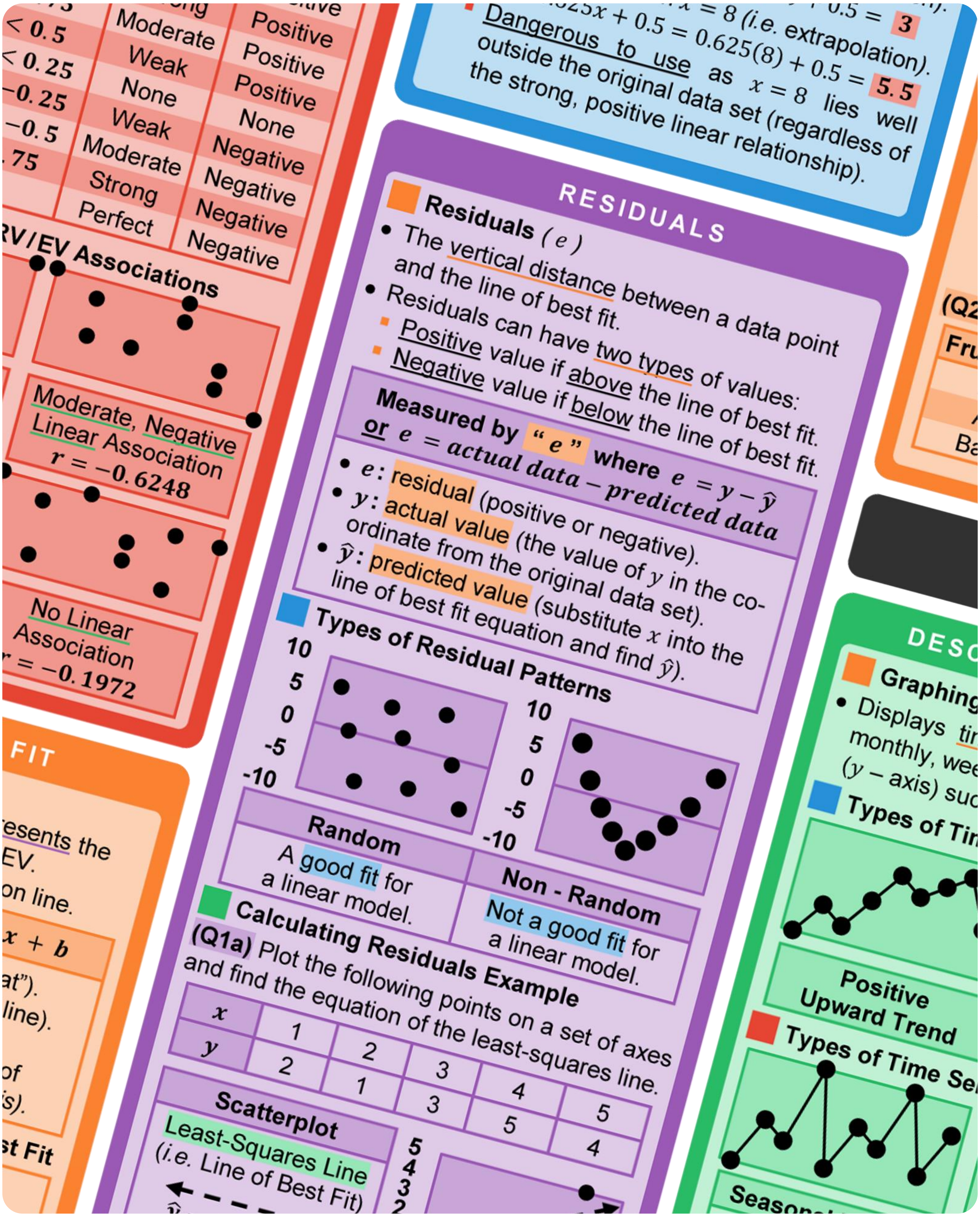


ATAR Mathematics Applications Units 3 & 4

Exam Notes for Western Australian Year 12 Students

Created by Anthony Bochrinis

Version 2.1 (Updated 27/09/18)

Sharpened®

ATAR Mathematics Applications Units 3 & 4 Exam Notes

Created by Anthony Bochrinis

Version 2.0 (Updated 12/07/18)
A Sharpened® Resource

► About the Creator - Anthony Bochrinis

Hello! My name is Anthony and I graduated from high school in 2012, completed a Bachelor of Actuarial Science in 2015, completed my Graduate Diploma in Secondary Education in 2017 and am now a secondary mathematics teacher!

My original exam notes (created in 2013) were inspired by Severus Snape's copy of Advanced Potion Making in Harry Potter and the Half-Blood Prince; a textbook filled with annotations containing all of the pro tips and secrets to help gain a clearer understanding.

Thank you for being a part of my journey in realising that teaching is my lifelong vocation!

► Keep Up to Date with Sharpened®

To be kept in the loop with all the latest resource releases, seminar dates, updates, news and tips, be sure to follow us on our social media and bookmark our website!

Instagram
@sharpenedau

Facebook
@sharpenedresources

Website
sharpened.com.au

2655		**5**	**44**
Total Creation Time (in Minutes)		Major Topics	Minor Topics
7472	**42**	**98**	**36**
Words	Formulae	Examples	Diagrams

► How to Use these Exam Notes

These exam notes are designed to be a complement to your studies throughout the year. As such, I recommend using these exam notes during class, during tests, whilst studying at home or in the library and even in the calculator-assumed section of your mock and WACE exams.

These exam notes are entirely yours, and under our licensing information, you are free to edit or modify these notes according to your liking, so long as they are not distributed!

The layout of these notes follows a simple system under our new design format:

- **Exam Notes Layout:** there are four pages each laid out in four columns.
- **Major Topics:** are coloured in black and act as a divider for the content that follows.
- **Minor Topics:** are smaller topics that are written at the top of each coloured box.
- **Minor Topic Subheadings:** within each minor topic are subheadings that are headed with a coloured square. This acts as a further segmentation of the exam notes which makes finding specific content that you are looking for much more straightforward.

★ NOTE: When you print these exam notes, please print pages 5 – 8 double sided ★

► FAQ – Frequently Asked Questions

FAQ #1: Are these exam notes even relevant according to the SCSA syllabus?

Our ATAR mathematics exam notes/cheat sheets are designed to be used by Western Australian year 12 students in preparation for their mock and WACE exams. As such, all of the content found in our exam notes are based on the official School Curriculum and Standards Authority (SCSA) syllabus.

FAQ #2: Am I allowed to use these exam notes in my Mock and WACE Exams?

In short, yes! Our exam notes are best used in the calculator assumed section of your mock and WACE exams. However, if you're worried that our exam notes will be confiscated in your exams or if you think it'll look weird if all of your classmates have the same notes, don't worry!

In 2016, we had over 8,000 WA year 12 students take our notes into their WACE exam. This didn't go unnoticed! As a result, the School Curriculum and Standards Authority (SCSA) have added a rule as of the 2017 Year 12 Handbook that allows any commercial notes to be used in WACE exams!

► FAQ – Frequently Asked Questions Continued

FAQ #3: Do these exam notes get updated at all?

Yes! We constantly receive advice for changes by teachers from high schools all across Western Australia! We use this expert knowledge to provide you with the most up to date set of exam notes possible! As such, please check back on our website before you sit any tests or exams as there may be an update that helps you earn you a few extra marks!

FAQ #4: Why are these exam notes free?

After realising that teaching is my life's goal, I want to spread the gift of mathematics and its underlying principles all around WA (and someday maybe even the world). To make it equally fair and accessible to everyone, I decided to make these notes free for everyone. If you like my notes and would like to support me, I would appreciate looking at the other paid services that Sharpened® has to offer.

► Exam Notes Disclaimer

Although material from the School Curriculum and Standards Authority is included as part of this Course Handbook, it does not constitute endorsement of this publication.

All effort is made to ensure that the information contained in these exam notes are correct and accurate but does not provide a guarantee to the authenticity of this content. By using these exam notes, you agree to the terms and conditions listed on the Sharpened® website.

These exam notes are designed to augment study and may not reflect what may be covered in class or school based examinations or assessments.

For any questions relating to any matter, please contact us at:

- **Website:** www.sharpened.com.au
- **E-Mail Address:** anthony@sharpened.com.au

A final word …

Thank you for downloading my exam notes and I really hope they help to sharpen your knowledge. Please take comfort in the fact that Sharpened® is here to support you outside of the classroom! You can do this and I wish you all the very best!

Now comes the fun part …

Without any further ado, I present to you my exam notes!

Yours Sincerely,
Anthony Bochrinis

BIVARIATE DATA

TYPES OF VARIABLES

Response and Explanatory Variables

- Response Variable (RV)
  - Also known as the dependent variable.
  - Plotted on the vertical axis ($y-axis$).
- Explanatory Variable (EV)
  - Also known as the independent variable.
  - Plotted on the horizontal axis ($x-axis$).

The Response Variable (RV) depends on the Explanatory Variable (EV)

Examples of RV's with Matching EV's

- The RV, weight loss (kg), depends on the EV, time spent dieting ($days$).
- The RV, wage ($dollars$), depends on the EV, time spent working ($hours$).
- The RV, heart rate (bpm), depends on the EV, caffeine consumption (mg).

CORRELATION COEFFICIENT

Pearson's Correlation Coefficient (r)

- Measures the direction and strength of a linear relationship between a RV and an EV.

Measured by "r" where $-1 \le r \le 1$

Coefficient of Determination (r^2)

- Calculated by squaring Pearson's correlation coefficient (*i.e.* $r^2 = r \times r$).

Measured by "$r^2 \times 100\%$" where $0 \le r^2 \le 1$

- r^2 measures the percentage variation in the RV with the variation in the EV.
  - For example, if $r^2 = 0.85$, then "85% of the variation in the RV can be explained by the variation in the EV".
- As r^2 increases, the regression line becomes a more appropriate model for the data.

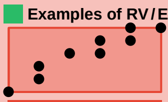
SCATTERPLOTS

Describing a Scatterplot

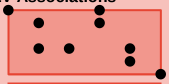
- Form: pattern type (*i.e.* linear or non-linear).
- Direction: where the points tend towards
  - Positive: from bottom-left to top-right
  - Negative: from top-left to bottom-right
- Strength: how closely the points follow a linear pattern (*e.g.* perfect, strong, etc.).

Value of r	Strength	Direction
$r = 1$	Perfect	Positive
$0.75 \le r < 1$	Strong	Positive
$0.5 \le r < 0.75$	Moderate	Positive
$0.25 \le r < 0.5$	Weak	Positive
$-0.25 \le r < 0.25$	None	None
$-0.5 \le r < -0.25$	Weak	Negative
$-0.75 \le r < -0.5$	Moderate	Negative
$-1 < r < -0.75$	Strong	Negative
$r = -1$	Perfect	Negative

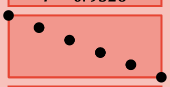
Examples of RV / EV Associations



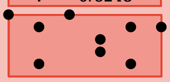
Strong, Positive Linear Association $r = 0.9526$



Moderate, Negative Linear Association $r = -0.6248$



Perfect, Negative Linear Association $r = -1$



No Linear Association $r = -0.1972$

LINE OF BEST FIT

Line of Best Fit ($\hat{y}$)

- A linear equation that best represents the relationship between a RV and EV (also known as the least-squares regression line).

Shown by "$\hat{y}$" where $\hat{y} = ax + b$

- $\hat{y}$: response variable (*a.k.a.* "y-hat").
- a: gradient (*i.e.* steepness of the line).
- x: explanatory variable.
- b: $y-$intercept (*i.e.* where the line of best fit intercepts/crosses the $y-axis$).

Interpreting Gradient of Line of Best Fit

$a > 0$	The EV increases by a for each increase in a single RV.
$a < 0$	The EV decreases by a for each increase in a single RV.

- For example, if $a = 0.95$, EV is amount of rainfall (mm) and RV is temperature (°C):
  - Therefore, the amount of rainfall increases by 0.95 mm for each increase in the temperature by 1 °C.

CORRELATION VS. CAUSATION

Correlation Does Not Imply Causation

- Regardless if there is a strong correlation coefficient between two variables, it does not provide sufficient evidence to prove that the two variables are causally related (*i.e.* that the EV actually causes the RV in reality).
- There are two reasons for this phenomenon:

Coincidence	Confounding
Caused by pure chance during data collection.	Caused by a third variable not being considered.

- To reduce the chance of these errors occurring, increase the data sample size.

Coincidence & Confounding Examples

- Coincidence: by pure chance, the collection of data comparing human height (m) and human arm length (m) may "accidently" show a strong negative correlation, where in reality this would certainly not be the case.
- Confounding: although amount of sunscreen applied (mL) and fainting (*# of cases*) may have a strong positive linear association, this association is likely due to a third variable, temperature (°C), not being considered.

PREDICTING DATA

Interpolation (*i.e. Predicting Inner Data*)

- Using the line of best fit to predict values that lie within the range of the original data.
- Safe to use, only if the association between the RV and EV is strong.

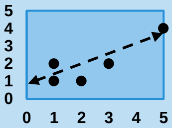
Extrapolation (*i.e. Predicting Outer Data*)

- Using the line of best fit to predict values that lie outside the range of the original data.
- Increasingly dangerous to predict data the further it is outside the current data. This is because the future is still unclear (regardless of how strong the correlation coefficient is).

Interpolation & Extrapolation Examples

Line of Best Fit $\hat{y} = 0.625x + 0.5$

Correlation Coefficient $r = 0.8539$



(Q1a) Predict $\hat{y}$ when $x = 4$ (*i.e.* interpolation).

- $\hat{y} = 0.625x + 0.5 = 0.625(4) + 0.5 = 3$

(Q1b) Predict $\hat{y}$ when $x = 8$ (*i.e.* extrapolation).

- $\hat{y} = 0.625x + 0.5 = 0.625(8) + 0.5 = 5.5$
- Dangerous to use as $x = 8$ lies well outside the original data set (regardless of the strong, positive linear relationship).

RESIDUALS

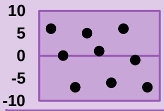
Residuals (e)

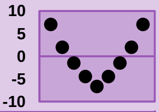
- The vertical distance between a data point and the line of best fit.
- Residuals can have two types of values:
  - Positive value if above the line of best fit.
  - Negative value if below the line of best fit.

Measured by "e" where $e = y - \hat{y}$ or $e = actual\ data - predicted\ data$

- e: residual (positive or negative).
- y: actual value (the value of y in the co-ordinate from the original data set).
- $\hat{y}$: predicted value (substitute x into the line of best fit equation and calculate $\hat{y}$).

Types of Residual Patterns





Random	Non - Random
A good fit for a linear model.	Not a good fit for a linear model.

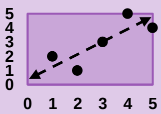
Calculating Residuals Example

(Q1a) Plot the following points on a set of axes and find the equation of the least-squares line.

x	1	2	3	4	5
y	2	1	3	5	4

Scatterplot

Least-Squares Line (*i.e.* Line of Best Fit) $\hat{y} = 0.8x + 0.6$

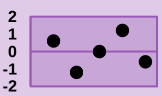


(Q1b) Find, and comment on, the residual plot.

$\hat{y}$	1.4	2.2	3	3.8	4.6
e	0.6	-1.2	0	1.2	-0.6

Residual Plot

Random pattern. Therefore, a linear model is a good fit.



TWO – WAY TABLES

Percentage Two-Way Tables

- Created using two different methods:
  - Using row sums (row percentages)
  - Using column sums (column percentages)

Two-Way Tables Example

(Q1) A survey asking seniors if they want increased healthcare increase is shown:

	Agree	Disagree	Undecided
Under 60	16	20	14
60 – 65	48	24	8
Over 60	40	7	3

(Q1a) What is the explanatory variable?

- The RV depends on the EV, hence the EV is Age (*i.e.* decision depends on age).

(Q1b) A two-way table of the data above using row percentages is shown. Show how the percentages of **(A)** and **(B)** were calculated.

	Agree	Disagree	Undecided
Under 60	32% **(A)**	40%	28%
60 – 65	60%	30% **(B)**	10%
Over 60	80%	14%	6%

- Calculating **(A)**: find row sum of "Under 60" "Under 60" row sum $= 16 + 20 + 14 = 50$

$$(A) = \frac{16}{50} \times 100 = \frac{32}{100} \times 100 = 32\%$$

- Calculating **(B)**: find row sum of "60 – 65" "60 – 65" row sum $= 48 + 24 + 8 = 80$

$$(B) = \frac{24}{80} \times 100 = \frac{3}{10} \times 100 = 30\%$$

(Q1c) Comment on the association between the variables of age and decision.

- As age increases the percentage of seniors who agree increases, disagree decreases and are undecided decreases.

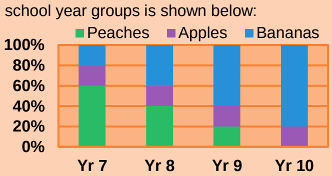
STACKED COLUMN GRAPH

Constructing Stacked Column Graphs

- To calculate each RV, find how tall each column is by comparing it to the $y-$axis.

Stacked Column Graph Example

(Q1) The favourite fruits of four different high school year groups is shown below:



(Q1a) Construct a two-way table from the data.

	Year 7	Year 8	Year 9	Year 10
Peach	60%	40%	20%	0%
Apple	20%	20%	20%	20%
Banana	20%	40%	60%	80%

(Q1b) Comment on noticeable associations.

- As student age increases, the percentage of peaches being popular decreases, apples being popular stays constant and bananas being popular increases.

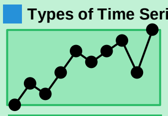
TIME SERIES

DESCRIBING TIME SERIES

Graphing Time Series

- Displays time (x – axis) such as annual, monthly, weekly or daily and another variable (y – axis) such as cost, sales or rainfall.

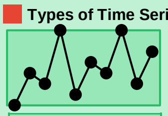
Types of Time Series Trends



Positive Upward Trend

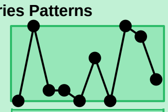
Negative Downward Trend

Types of Time Series Patterns



Seasonal Pattern

Peaks and troughs with similar sizes at regular intervals



Cyclical Pattern

Peaks and troughs with different sizes at irregular intervals

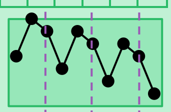
Time Series Example

(Q1) Plot the following time series and find an appropriate seasonal pattern in the data.

$Year$	1	2	3	4	5	6	7	8	9	10
$Cost$	4	7	6	3	6	5	2	5	4	1

Finding a Pattern

A seasonal pattern occurs after every 3 years (periods).



ODD MOVING AVERAGES

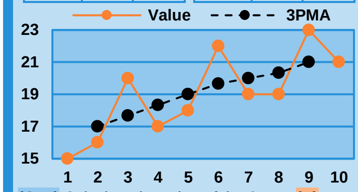
Moving Averages for Odd Patterns

$$3PMA = (a + b + c) \div 3$$

- $3PMA$: 3 point moving average.
- a - c: 3 successive data points associated with the 3PMA that you are looking for.

Calculating 3PMA Example

Period	Value	3PMA	Period	Value	3PMA
1	15	-	6	22	19.67
2	16	17 **(A)**	7	19	20
3	20	17.67	8	19	20.33
4	17	18.33	9	23 **(B)**	21
5	18	19	10	21	-



(Q1a) Calculate the value of the 3PMA **(A)**:

- Use 3 values that has **(A)** vertically halfway:

$$(A) = 3PMA_{t=2} = \frac{15 + 16 + 20}{3} = 17$$

(Q1b) Calculate the value of **(B)**:

- Use a group of 3 numbers that includes **(B)**:

$$\frac{19 + (B) + 23}{3} = 20.33 \qquad (B) + 42 = 61$$
$$19 + (B) + 23 = 61 \qquad (B) = 19$$

EVEN MOVING AVERAGES

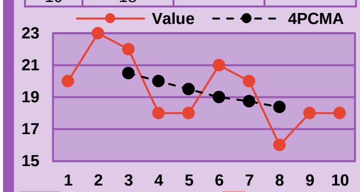
Moving Averages for Even Patterns

$$4PMA = (a + b + c + d) \div 4$$

- $4PMA$: 4 point moving average.
- a - d: 4 successive data points associated with the 4PMA that you are looking for.

Calculating 4PCMA Example

Period	Value	4PMA	4PCMA
1	20	-	-
1.5	-	-	-
2	23 **(A)**	-	-
2.5	-	20.75	-
3	22	-	20.5
3.5	-	20.25	-
4	18	-	20
4.5	-	19.75 **(B)**	-
5	18	-	19.5
5.5	-	19.25	-
6	21	-	19
6.5	-	18.75	-
7	20	-	18.75
7.5	-	18.75	-
8	16	-	18.375 **(C)**
8.5	-	18	-
9	18	-	-
9.5	-	-	-
10	18	-	-



(Q1a) Calculate the value of **(A)**:

- Use a group of 4 numbers that includes **(A)**:

$$\frac{20 + (A) + 22 + 18}{4} = 20.75 \qquad (A) + 60 = 83$$
$$20 + (A) + 22 + 18 = 83 \qquad (A) = 23$$

(Q1b) Calculate the value of the 4PMA **(B)**:

- Use 4 values that has **(B)** vertically halfway:

$$(B) = 4PMA_{t=4.5} = \frac{22 + 18 + 18 + 21}{4} = 19.75$$

(Q1c) Calculate the value of the 4PCMA **(C)**:

- Average the 2 closest 4PMA values to **(C)**:

$$(C) = 4PCMA_{t=8} = \frac{18.375 + 18}{2} = 18.375$$

Calculating 4PCMA Shortcut Formula

$$4PCMA = (0.5a + b + c + d + 0.5e) \div 4$$

- $4PCMA$: 4 point centered moving average.
- a - e: 5 successive data points associated with the 4PCMA that you are looking for.

(Q1d) Use this formula to find the 4PCMA **(C)**:

- As this is a 4PCMA, find the 5 numbers in the value column that has **(C)** vertically halfway:

$$(C) = \frac{0.5 \times 18 + 21 + 20 + 16 + 18 + 0.5 \times 18}{4}$$
$$= \frac{9 + 21 + 20 + 16 + 18 + 9}{4} = 18.75$$

ATAR Math Applications
Units 3 & 4 Exam Notes

Created by **Anthony Bochrinis**

DESEASONALISING DATA

Deseasonalising Data

- Smoothing the data to reduce the presence of seasonal fluctuations (e.g. reduce the fluctuations in apple picking between summer and winter months of each year).

Step 1	Calculate the average of each of the non-seasons. Non-seasons are typically years, months or weeks.
Step 2	Divide each of the original data values respectively by the average of the non-seasons found in Step 1.
Step 3	Finding Seasonal Indices: Calculate the average of each of the seasons found in Step 2. Seasons are commonly time periods, days or seasons (summer, winter, autumn, spring) of the year.
Step 4	Deseasonalising the Data: Divide each of the original data values respectively by the seasonal indices found in Step 3.

Deseasonalising Data Example

(Q1a) Deseasonalise the following data set:

Yr / Pd	Period 1	Period 2	Period 3
Year 1	411	648	699
Year 2	532	632	741

- Step 1: Average non-seasons (*i.e.* years).

Average Year 1	$\frac{411+648+699}{3} = 586$
Average Year 2	$\frac{532+632+741}{3} = 635$

- Step 2: Divide original data by Step 1.

Yr / Pd	Period 1	Period 2	Period 3
Year 1	$\frac{411}{586} = 0.7014$	$\frac{648}{586} = 1.1058$	$\frac{699}{586} = 1.1928$
Year 2	$\frac{532}{635} = 0.8378$	$\frac{632}{635} = 0.9953$	$\frac{532}{635} = 1.1669$

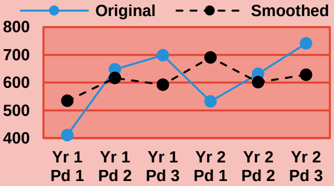
- Step 3: Find seasonal index by averaging seasons in Step 2 (*i.e.* periods).

Average Period 1	$\frac{0.7014+0.8378}{2} = 0.7696$
Average Period 2	$\frac{1.1058+0.9953}{2} = 1.0505$
Average Period 3	$\frac{1.1928+1.1669}{2} = 1.1799$

- Step 4: Deseasonalise data by dividing the original data respectively by Step 3.

Yr / Pd	Period 1	Period 2	Period 3
Year 1	$\frac{411}{0.7696} = 534$	$\frac{648}{1.0505} = 617$	$\frac{699}{1.1799} = 582$
Year 2	$\frac{532}{0.7696} = 691$	$\frac{632}{1.0505} = 602$	$\frac{532}{1.1799} = 628$

(Q1b) Graph the deseasonalised data:



SEASONAL INDICES

Properties of Seasonal Indices

Percentage Property (%)
Converting each of the seasonal indices to percentages indicates performance above or below average for that season.

Additive Property (+)
The sum of seasonal indices equals the number of seasons in the original data set.

Seasonal Indices Examples

(Q1) The deseasonalised data below shows a pool businesses' revenue over two years:

Yr / Qr	Qr 1	Qr 2	Qr 3	Qr 4
Year 1	11.5	11.7	10.8	10.9
Year 2	13.0	12.7	13.8	13.6
Seasonal Index	1.308 **(A)**	0.942	0.651 **(B)**	1.099 **(C)**

(Q1a) How is seasonal index **(A)** is calculated?

- As there are 4 quarters (*i.e.* seasons), the seasonal indices must add up to 4:
  $(A) + 0.942 + 0.651 + 1.099 = 4$
  $(A) = 4 - 2.692,\ (A) = \mathbf{1.308}$

(Q1b) Interpret seasonal index **(B)**:

- In Quarter 3 (Jul, Aug, Sep), revenue is expected to be **34.9%** below average:

$(B) = 0.651 \rightarrow 65.1\% \rightarrow 34.9\%$ below average

► Topic Is Continued In Next Column ◄

SEASONAL INDICES

(Q1c) Interpret seasonal index **(C)**:

- In Quarter 4 (Oct, Nov, Dec), revenue is expected to be **9.9%** above average:

$(C) = 1.099 \rightarrow 109.9\% \rightarrow 9.9\%$ above average

(Q1d) Find the line of best fit of the data:

- Using calculator: $\hat{y} = \mathbf{0.393x + 10.482}$

(Q1e) Predict the revenue for Qr 3, Year 4:

- Quarter 3, Year 4 is at time $= 11$
- $\hat{y} = 0.393(11) + 10.482 = 14.805$

14.805 is the deseasonalised value for Qr 3, Year 4, therefore we need to multiply by the seasonal index for Qr 3 to seasonalise it:

$14.805 \times 0.651 = \mathbf{9.638}$

SEQUENCES

TYPES OF SEQUENCES

Sequence Notation

- T_n: the value of the n^{th} term in the sequence.
- a: the value of the first (initial) term in the sequence (i.e. the value of T_1).
- d: common difference between each term (note: arithmetic sequences only).
- r: common ratio between each term (note: geometric sequences only).

Arithmetic Sequences (+ or −)

- Each term is found by adding or subtracting a constant to or from the previous term.
- a.k.a. arithmetic progression (AP).

4 →(+3) 7 →(+3) 10 →(+3) 13 →(+3) 17

$d = T_{n+1} - T_n$ or $d = T_2 - T_1$

Geometric Sequences (× or ÷)

- Each term is found by multiplying or dividing a constant to or from the previous term.
- a.k.a. geometric progression (GP).

0.5 →(× 4) 2 →(× 4) 8 →(× 4) 32 →(× 4) 128

$r = T_{n+1} \div T_n$ or $r = T_2 \div T_1$

Explicit and Recursive Formulae

- Explicit: finds the value of any term.
- Recursive: finds the next term in the sequence if the previous term is known.
  - Note: T_1 must also be stated with the rule.

Arithmetic Sequences (AP)	
Explicit	$T_n = a + (n-1) \times d$
Recursive	$T_{n+1} = T_n + d,\ T_1 = a$
Geometric Sequences (GP)	
Explicit	$T_n = a \times r^{n-1}$
Recursive	$T_{n+1} = T_n \times r,\ T_1 = a$

SEQUENCES EXAMPLES

Arithmetic Sequences Examples

(Q1) Consider the sequence: $12, 7, 2, -3 \ldots$ Find which term of the sequence is -118

- Explicit, $a = 12,\ d = 7 - 12 = -5$

$T_n = 12 + (n-1) \times (-5) = -5n + 17$
$-118 = -5n + 17,\ \ -135 = -5n,\ \ n = \mathbf{27}$

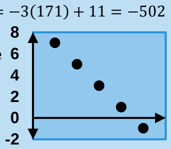
(Q2) Let $T_{n+1} = T_n - 3$ and $T_4 = 9$, find T_1

$T_1 = T_4 + 3 + 3 = 9 + 3 + 3 = \mathbf{18}$

(Q3) Let $T_n = -3n + 11$, find the first term in the sequence that is less than -500.

$-500 = -3n + 11,\ 3n = 511,\ n = 170.33$
$\therefore$ **171**$^{\text{st}}$ **term** as $T_{171} = -3(171) + 11 = -502$

(Q4) Graph the first 5 terms of the sequence $T_{n+1} = T_n - 2,\ T_3 = 7$ and comment on the shape of the graph:



Linear, decreasing

Geometric Sequences Examples

(Q1) Consider the sequence: $256, 128, 64, 32$ Write the recursive rule for this sequence:

- Recursive, $a = 256,\ r = 128 \div 256 = 0.5$

$T_{n+1} = \mathbf{0.5T_n},\ \mathbf{T_1 = 256}$

(Q2) Find when sequence $6, 30, 150, 750 \ldots$ first becomes larger than 100,000:

- Explicit, $a = 6,\ r = 30 \div 6 = 5$

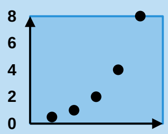
Using calculator, solve $100000 = 6 \times 5^n$ for n
$n = 6.04 \therefore$ **7**$^{\text{th}}$ **term** as $T_7 = 6 \times 5^7 = 468{,}750$

(Q3) Find the 10$^{\text{th}}$ term of the sequence: $2, 2.4, 2.88, 3.456 \ldots$ to two decimal places:

- Explicit, $a = 2,\ r = 2.4 \div 2 = 1.2$

$T_n = 2 \times 1.2^{n-1},\ T_{10} = 2 \times 1.2^9 = \mathbf{6.19}$

(Q4) Graph the first 5 terms of the sequence $T_{n+1} = T_n \times 2,\ T_2 = 0.5$ and comment on the shape of the graph:



Exponential, increasing

GROWTH AND DECAY

Exponential Growth and Decay

- Exponential growth/decay is a type of GP.

Growth (+) Formulae	
Explicit	$T_n = a \times (1 + r)^t$
Recursive	$T_{n+1} = (1 + r) \times T_n,\ T_1 = a$
Decay (−) Formulae	
Explicit	$T_n = a \times (1 - r)^t$
Recursive	$T_{n+1} = (1 - r) \times T_n,\ T_1 = a$

Exponential Growth/Decay Examples

(Q1) Write a recursive rule to model 8 rabbits growing in population at a rate of 40% per year.

- Recursive, $a = 8,\ r = 40\% = 0.4$

$T_{n+1} = (1 + 0.4) \times T_n,\ \mathbf{T_{n+1} = 1.04T_n,\ T_1 = 8}$

(Q2) Write a rule to show the area of a $350m^2$ oil slick that reduces by 6% every hour.

- Explicit, $a = 350,\ r = 6\% = 0.06$

$T_n = 350 \times (1 - 0.06)^n,\ \mathbf{T_n = 350 \times 0.94^n}$

RECURRENCE RELATIONS

First-Order Recurrence Relation

- A combination of AP and GP sequences:

$$T_{n+1} = r \times T_n + d,\ T_1 = a$$

- 3 types of first-order recurrence relations:
  - Long-term increasing solution (as n increases, T_n increases indefinitely).
  - Long-term decreasing solution (as n increases, T_n decreases indefinitely).
  - Long-term steady state solution (as n increases, T_n approaches a certain value).

Recurrence Relation Example

(Q1) Comment on the long-term solution of the recurrence relation, $T_{n+1} = 5T_n - 2,\ T_1 = 1$

n	1	2	3	4	5
T_n	1	3	11	53	263

This has a long-term **increasing solution**.

Finding a Steady State Solution

Algebra Method	
Step 1	Substitute both T_{n+1} and T_n with T. Note: ignore initial value T_1.
Step 2	Rearrange equation and solve for T. This is the steady state solution.
Calculator Method	
Step 1	Using sequences app, analyse terms for large values of n (*e.g.* T_{50}) and find the steady state solution.

(Q1) Comment on the long-term solution of the recurrence relation, $T_{n+1} = 0.5T_n + 3,\ T_1 = 4$

(Q1a) Find the long-term steady state solution of this sequence using an algebraic method.

$T_{n+1} = 0.5T_n + 3 \quad T - 0.5T = 3 \quad T = 3 \div 0.5$
$T = 0.5T + 3 \quad 0.5T = 3 \quad \mathbf{T = 6}$

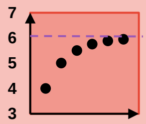
(Q1b) Find the long-term steady state solution of this sequence by graphing the terms.

n	1	2	3	4	5
T_n	4	5	5.5	5.75	5.875

… Continued for large values of n …

n	46	47	48	49	50
T_n	5.998	5.999	5.999	6	6

Steady State
As n increases, T_n approaches the steady state value of **6** and is shown by the graph:



FINANCE

SIMPLE INTEREST

Simple Interest Formula

- Interest earned/owed is constant over time.
- Simple interest pattern is linear.

$I = P \times R \times T$	$A = I + P$

- A: total amount (principal plus interest).
- P: principal (starting amount).
- I: total interest earned/ owed.
- R: interest rate (as a decimal).
- T: time (must be converted to years).

Simple Interest Example

(Q1) Ellie purchased a mobile phone worth $600 using her credit card that charges 19.8% p.a. simple interest on the 30$^{\text{th}}$ of March. She paid the account on the 11$^{\text{th}}$ of April.

(Q1a) What was the total interest charged?

$I = PRT = 600 \times 0.198 \times \frac{13}{365} = \mathbf{\$4.23}$

(Q1b) What is the total amount that Ellie paid for the mobile phone?

$A = I + P = 4.23 + 600 = \mathbf{\$604.23}$

COMPOUND INTEREST

Compound Interest Formula

- Interest earned/owed increases over time.
- Compound interest pattern is exponential.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$	$I = A - P$

- A: total amount (principal plus interest).
- P: principal (initial/starting amount).
- I: total amount of interest earned/owed.
- r: annual interest rate (as a decimal).
- n: number of times in which interest is compounded per year (see table below).
- t: time (must be converted to years).

Effective Annual Rate of Interest

- Effective annual rate of interest converts $i\%\ p.a.$ compounding n times per year into $i_{effective}\%\ p.a.$ compounding annually.

$$i_{effective} = \left(1 + \frac{i}{n}\right)^n - 1$$

- $i_{effective}$: effective annual rate of interest (as a decimal).
- i: annual interest rate (as a decimal).
- n: number of times in which interest is compounded per year (see table below).

Frequency of Compounding Interest

- The more times interest is compounded per year, the more interest is earned.
- The higher the value of n, the higher the effective annual rate of interest.
- There is diminishing returns (*i.e.* a limit) on the amount of interest gained as n increases.

n	Freq.	i	Converting to effective annual rate →	$i_{effective}$
Yearly	1	5%		5%
Six-Monthly	2	5%		5.062%
Quarterly	4	5%		5.095%
Monthly	12	5%		5.116%
Fortnightly	26	5%		5.122%
Weekly	52	5%		5.125%
Daily	365	5%		5.127%

FORMS OF COMPOUND INTEREST

Compound Interest Recurrence Relation

$$A_{n+1} = \left(1 + \frac{i}{n}\right) \times A_n + r,\ \ A_o = P$$

- i: interest rate (as a decimal).
- n: number of times in which interest is compounded per year (see table above).
- r: regular payments or contributions:
  - For investments, r is positive
  - For loans and annuities, r is negative
- P: principal (initial/starting amount).

Compound Interest Financial Calculator

N	Number of Periods (P/Y × # Years)
I%	Annual Interest Rate
PV	Present Value
PMT	Regular Payment or Contribution
FV	Future Value
P/Y	Number of Payments Per Year
C/Y	Number of Times Interest is Compounded Per Year

TYPES OF FINANCIAL PRODUCTS

Loans

- Borrowing a sum of money that needs to be paid back in full over a certain period of time.

PV	PMT	FV
Positive	Negative	0

Investments

- Depositing money into the bank that grows over time due to interest, whilst making regular contributions to grow the account.

PV	PMT	FV
Negative	Negative	Positive

Annuities

- A sum of money that is withdrawn from the bank in regular intervals until it fully depletes.

PV	PMT	FV
Negative	Positive	0

Perpetuities

- Withdrawing money from an account that is equal to the amount of interest earned so that the principal amount never depletes.

$$Q = PE$$

- Q: annual withdrawal amount.
- P: principal (initial/starting amount).
- E: effective annual interest rate (decimal)

ATAR Math Applications
Units 3 & 4 Exam Notes

 Version: **2.1**

COMPOUND INTEREST FORMULA

Compound Interest and Effective Rates

(Q1) $50,000 is invested into a bank with a rate of 7.67% p.a., compounding half-yearly over 3 years. How much interest does it accrue?

$$A = 50000\left(1+\frac{0.0767}{2}\right)^{2\times3} = \$62666.09$$

$$I = A - P = 62666.09 - 50000 = \$12666.09$$

(Q2) $30,000 can be invested into a bank using two different saving schemes:

- **X**: 6.22% p.a. compounding monthly
- **Y**: 6.25% p.a. compounding quarterly

Find the effective rate of interest for both schemes, which would pay more after 3 years?

$$X = \left(1+\frac{0.0622}{12}\right)^{12} - 1 = 0.0640 = 6.4\%$$

$$Y = \left(1+\frac{0.0625}{4}\right)^{4} - 1 = 0.06398 = 6.398\%$$

∴ **Scheme X pays more interest than Y**

(Q3) Lucy wants to set up a perpetuity of $5,000 per year at a bank that pays 6% p.a. compounding monthly. Determine how much is required to maintain this perpetuity.

- For a perpetuity, $Q = P \times E$

$$Q = 5000,\ E = \left(1+\frac{0.06}{12}\right)^{12} - 1 = 0.06168$$

$$Q = P \times E,\ 5000 = P \times 0.06168$$

$$P = 5000 \div 0.06168,\ P = \$81,066.43$$

COMPOUND INTEREST TABLES

Compound Interest Table Form

(Q1) Sophia borrows $500 at 6% p.a. compounding quarterly and makes quarterly payments of $150 to pay off the loan.

- Financial product type: **loan**
- Regular payment: **payment** (negative value)
- Quarterly I%: $i \div n = 6 \div 4 = 1.5\% = 0.015$

Quarter	1	2	3
Start Amount	$500	$357.50	$212.86
Interest	+ $7.50	+ $5.36	+ $3.19
Payment	- $150	- $150	- $150
End Amount	$357.50	$212.86	$66.05

(Q2) Lucas invests $600 into an account that pays 4% p.a. compounding monthly and makes monthly deposits of $50.

- Financial product type: **investment**
- Regular payment: **deposit** (positive value)
- Monthly I%: $i \div n = 4 \div 12 = 0.\dot{3}\% = 0.00\dot{3}$

Month	1	2	3
Start Amount	$600	$652	$704.17
Interest	+ $2	+ $2.17	+ $2.35
Deposit	+ $50	+ $50	+ $50
End Amount	$652	$704.17	$756.52

(Q3) Charlotte invests $1,000 into an annuity that pays $250 every six months at 8% p.a. compounding half-yearly.

- Financial product type: **annuity**
- Regular payment: **withdraw** (positive value)
- Half-yearly I%: $i \div n = 8 \div 2 = 4\% = 0.04$

Month	1	2	3
Start Amount	$1,000	$790	$571.60
Interest	+ $40	+$31.60	+$22.86
Withdraw	- $250	- $250	- $250
End Amount	$790	$571.60	$344.46

RECURSIVE RELATIONS

Compound Interest Recursive Rules

(Q1) Oliver has borrowed $8,750 to buy a car and is making repayments of $750 at the end of each month on the loan, with interest charged monthly. The interest for the first month totaled to $65.50.

(Q1a) Calculate the annual interest rate.

- $r = i \div P,\ r = 65.5 \div 8750,\ r = 0.007486$
- ∴ Monthly interest rate is 0.07486%
- Annual rate $= 0.007486 \times 12 = 0.08983$
- ∴ Annual interest rate is **8.98%**

(Q1b) Express this loan as a recursive rule.

$$T_{n+1} = \left(1+\frac{0.0898}{12}\right)T_n - 750,\ T_0 = 8750$$

$$T_{n+1} = 1.007486T_n - 750,\ T_0 = 8750$$

(Q1c) How much does he owe after 1 year?

1 year = 12 months, $T_{12} = \$932.26$

(Q1d) How much interest in total is charged after the first year of the loan?

- Total paid off loan $= 8750 - T_{12}$ $= 8750 - 932.26 = \$7,817.74$
- Total repayments $= 750 \times 12 = \$9,000$
- To calculate interest, subtract the amount paid off loan from the total repayments $= 9000 - 7817.74 = \$1,182.26$

FINANCIAL CALCULATOR

Compound Interest Financial Calculator

(Q1) Jackson borrows $20,000 at 12% p.a. compounding monthly. He pays $350 every month to pay off the loan. How much would he still owe after 5 years of payments?

Type	Loan	PMT	−350
N	5 × 12 = 60	**FV**	**−$7749.55**
I%	12	**P/Y**	12
PV	20000	**C/Y**	12

(Q2) Lily invests $10,000 at 7% p.a. compounding half-yearly. Lily wants her account to reach $50,000 in 10 years. How much does she need to deposit every six months to reach this goal?

Type	Investment	PMT	**−$1064.44**
N	2 × 10 = 20	**FV**	50000
I%	7	**P/Y**	2
PV	−10000	**C/Y**	2

(Q3) Emily borrows $25,000 at a rate of 12% p.a. compounding half-yearly. Her loan needs to be repaid in 4 years. What would Emily's half-yearly repayments need to be?

Type	Loan	PMT	**−$4025.90**
N	2 × 4 = 8	**FV**	0
I%	12	**P/Y**	4
PV	25000	**C/Y**	4

(Q4) Grace invests $700,000 to buy an annuity that pays $50,000 at 5.4% p.a. compounding annually. How many years will Grace be able to withdraw money for?

Type	Annuity	PMT	50000
N	**26.82 = 27**	**FV**	0
I%	5.4	**P/Y**	1
PV	−700000	**C/Y**	1

(Q5) Brandon wants to save $100,000 in 5 years' time and begins by making a $1,000 deposit. If a bank offers 6% p.a. compounding monthly, how much does Brandon need to deposit each quarter to reach his goal?

Type	Investment	PMT	**−$4263.09**
N	4 × 5 = 20	**FV**	100000
I%	6	**P/Y**	4
PV	−1000	**C/Y**	12

ANALYSING LOANS

Calculating Final Payment

- When paying off loans, the final payment will often be less than the regular payment.
- Two formulae that calculates final payment:

$$Final = r + T_{FN} \qquad Final = T_{LP} \times \left(1+\frac{i}{n}\right)$$

- $Final$: final payment amount.
- T_{FN} : the first negative value that appears in the recurrence relation table.
- T_{LP} : the last positive value that appears in the recurrence relation table.
- r : regular payment amount.
- i : annual interest rate (as a decimal).
- n : number of times in which interest is compounded per year.

Calculating Total Cost of a Loan

- Total cost of loan adds the sum of the regular payments (based on number of regular payments) to the irregular final payment.

$$Total\ Cost = (n \times r) + Final$$

- n : number of full payments.
- r : regular payment amount.
- $Final$: final payment amount.

Complex Loans Example Question

(Q1) Jordan takes out a $12,000 loan to purchase his first car. The bank offered a loan at a rate of 2.5% p.a. compounding monthly. Jordan makes monthly repayments of $500.

(Q1a) Express this loan as a recursive rule.

$$T_{n+1} = 1.025T_n - 500,\ T_0 = 12000$$

(Q1b) How long does it take to pay the loan?

$T_{26} = T_{FN} = -176.67$ ∴ **26 months**

(Q1c) Calculate the final payment amount.

- Recurrence relation table, scrolled down:

T_{24}	820.9459	Irrelevant Value
T_{25}	322.6562	Last Positive Value (T_{LP})
T_{26}	-176.6716	First Negative Value (T_{FN})
T_{27}	-677.0396	Irrelevant Value

- Use 1st formula: $Final = r + T_{FN}$

$Final = r + T_{26} = 500 - 176.6716 = \$323.33$

- Or use 2nd formula: $Final = T_{LP} \times \left(1+\frac{i}{n}\right)$

$$Final = T_{25} \times \left(1+\frac{0.025}{12}\right) = \$323.33$$

(Q1d) Calculate the total cost of the car.

$$Total\ Cost = (n \times r) + Final$$
$$Total\ Cost = (25 \times 500) + 323.33$$
$$= \$12823.33$$

GRAPHS & NETWORKS

EULER AND PLANAR GRAPHS

Euler's Rule to Verify Planar Graphs

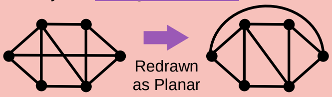
- Euler's Rule only works on planar graphs:

$$V - E + F = 2$$

- V : number of vertices (a.k.a. nodes).
- E : number of edges (a.k.a. arcs).
- F : number of faces (a.k.a. regions).

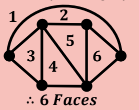
Sketching Planar Graphs

- A graph is planar if it can be redrawn in such a way that no edges cross over each other.



- When counting faces in a planar graph, the outside region counts as one face.

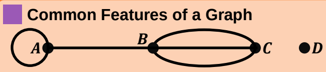
(Q1) Verify that Euler's rule works on the redrawn planar graph shown above.



$V = 6, E = 10, F = 6$
$V - E + F = 2$
$6 - 10 + 6 = 2$
$2 = 2$
∴ Euler's rule is verified.

NETWORK TERMINOLOGY

Common Features of a Graph

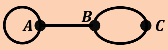


- Loop: edge joining a vertex to itself (e.g. A).
- Multiple edges: two or more edges that have the same start and end vertices (e.g. B-C).
- Isolated vertex: a disconnected vertex that is separate from the rest of the graph (e.g. D).
- Bridge: an edge that connects two parts of a graph that would otherwise result in an isolated vertex or vertices. (e.g. A-B).

Degree of a Vertex

- Number of edges connected to a vertex.
- The degree of a loop is counted twice.

(Q1) What is the degree of vertex A, B and C?



$A = 3,\ B = 3,\ C = 2$

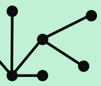
TYPES OF GRAPHS

Common Types of Graphs

- Simple Graph: a type of graph that does not contain loops or multiple edges.
- Connected Graph: a type of graph with a possible path between every vertex.
- Subgraph: a graph that has vertices and edges that are a subset of a larger graph.
- Directed Graph (Digraph): a graph where all edges are directed (shown by an arrow).
- Weighted Graph: a type of graph with edges that have been assigned a numerical weight.

Trees

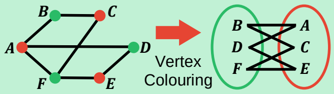
- Connected graph that does not contain any cycles or multiple edges.



Bipartite Graphs

- Graph that has two sets of vertices where any edges can only connect the two groups.
- The method of vertex colouring creates bipartite graphs from any graph:

Step 1	Using only two colours, colour all vertices such that no same colour is connected by one edge.
Step 2	Create a bipartite graph with a colour designated to each set.



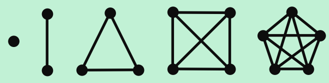
Complete Graphs (K_n)

- Graph with n vertices where every vertex is connected to all other vertices by one edge.

$$Number\ of\ edges\ in\ K_n = \frac{n(n-1)}{2}$$

- n: number of vertices in the graph.

K_n	Vertices	Edges	Faces	Planar
K_1	1	0	1	Yes
K_2	2	1	1	Yes
K_3	3	3	2	Yes
K_4	4	6	4	Yes
K_5	5	10	N/A	No



ADJACENCY MATRICES

Properties of an Adjacency Matrix

- Matrix that shows how many times each vertex is connected (adjacent) to another vertex by a single edge.
- From vertices on left, To vertices are above.
- Loops only count once in adjacency matrices.

Adjacency Matrix (Undirected Graph)

- Matrix is symmetrical along the diagonal.

From \ To	A	B	C
A	1	1	1
B	1	0	2
C	1	2	0

Adjacency Matrix (Directed Graph)

- Matrix is not symmetrical along the diagonal.

From \ To	A	B	C
A	1	0	0
B	1	0	1
C	1	1	0

TRANSITION MATRICES

Properties of a Transition Matrix

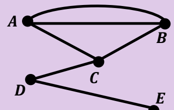
- Any entries in an adjacency matrix raised to the n^{th} power indicates how many ways it is possible to move to and from the points corresponding to that entry in n steps.

1-Step $= M$	2-Step $= M^2$
1-Step or 2-Step $= M + M^2$	

- M : adjacency matrix.
- M^2 : adjacency matrix squared.
- **1-Step** : a matrix showing number of ways to travel between vertices in 1 step.
- **2-Step** : a matrix showing number of ways to travel between vertices in 2 steps.
- **1 or 2-Step** : a matrix showing number of ways to travel between vertices in 1 or 2 steps (i.e. combines 1-Step and 2-Step).

Transition Matrix Example

(Q1) A connected graph and related transition matrices M, M^2 and $M + M^2$ are shown below:



$$M = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 5 & 1 & 2 & 1 & 0 \\ 1 & 5 & 2 & 1 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad M + M^2 = \begin{bmatrix} 5 & 3 & 3 & 1 & 0 \\ 3 & 5 & 3 & 1 & 0 \\ 3 & 3 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(Q1a) Explain how the highlighted entry "2" is calculated in transition matrix M.

- Matrix M shows 1-Step transitions.
- The highlighted entry shows that there are **2 ways** of going from **A to B in 1 step**.

(Q1b) Explain how the highlighted entry "0" is calculated in transition matrix M^2.

- Matrix M^2 shows 2-Step transitions.
- The highlighted entry shows that it is **impossible** to go from **E to D in 2 steps**.

(Q1c) Explain how the highlighted entry "3" is calculated in transition matrix $M + M^2$.

- $M + M^2$ shows 1- or 2-Step transitions.
- The highlighted entry shows that there are **3 ways** of going from **C to A in 1 or 2 steps**.

WALKS, PATHS AND TRAILS

Walks (Open and Closed)

- Walk: a sequence of vertices in a graph that represents a particular travel route.
- Open Walk: a type of walk that start and ends on two different vertices.
- Closed Walk: a type of walk that starts and ends on same vertex.
- Length of a Walk: number of edges of a walk.

Paths and Trails (Open and Closed)

- Paths and trails are types of walks that do not have to use all vertices and edges in a graph.

Name	Vertices	Edges
Open Path	Can't Repeat	Can't Repeat
Closed Path (a.k.a. cycle)	Can't Repeat*	Can't Repeat
Open Trail	May Repeat	Can't Repeat
Closed Trail (a.k.a. circuit)	May Repeat	Can't Repeat

- Exception to cycles*: only the starting and finishing vertex is allowed to repeat whilst all other vertices in the cycle cannot.

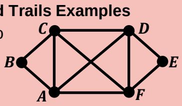
► Topic Is Continued On Next Page ◄

ATAR Math Applications
Units 3 & 4 Exam Notes

WALKS, PATHS AND TRAILS

Walks, Paths and Trails Examples

(Q1) Use the graph to write examples of all types of walks, paths and trails in the table:



Type of Walk, Path or Trail	Example
Open Walk	BACAFDE
Closed Walk	DCFACD
Open Path	BADEF
Closed Path (*a.k.a.* Cycle)	BCDEFAB
Open Trail	BCDFCA
Closed Trail (*a.k.a.* Circuit)	FACFDEF

EULERIAN GRAPHS

Eulerian/Semi-Eulerian Graphs

Eulerian Graphs

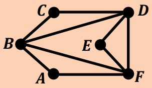
- Contains a closed trail (*i.e.* circuit) that visits all edges in the graph once only and may repeat vertices if needed.
- Every vertex has an even degree.
- Circuit starts/ends on the same vertex.

Semi-Eulerian Graphs

- Contains an open trail that visits all edges in the graph once only and may repeat vertices if needed.
- Contains one pair of vertices with an odd degree (all others have even degree).
- Trail starts and ends on either of the two odd-degree vertices.

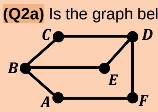
Eulerian Graph Examples

(Q1) Justify why the graph below is Eulerian.



This graph is Eulerian as all vertices have a degree of either 2 or 4, hence all **even degree**.

(Q2a) Is the graph below Semi-Eulerian?



This graph is Semi-Eulerian as only vertices **B and D** have **odd degrees**.

(Q2b) Where does the open trail start and finish in the semi-eulerian graph above?

- The trail will start/end on vertex **B and D**.

HAMILTONIAN GRAPHS

Hamiltonian/Semi-Hamiltonian Graphs

Hamiltonian Graphs

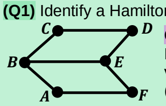
- Contains a closed path (*i.e.* cycle) that visits all vertices in the graph once only (except the start and end vertex) and does not have to pass through all edges.

Semi-Hamiltonian Graphs

- Contains an open path that visits all vertices in the graph once only and does not have to pass through all edges

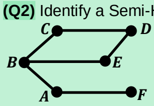
Hamiltonian Graph Examples

(Q1) Identify a Hamiltonian cycle below:



CDEFABC is a Hamiltonian cycle that visits all vertices once (except start/end).

(Q2) Identify a Semi-Hamiltonian path below:



CDEBAF is a Semi-Hamiltonian path as it must repeat a vertex in order to visit them all.

PRIM'S ALGORITHM

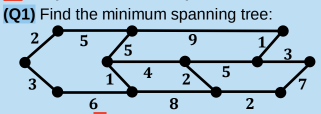
Prim's Algorithm (Graph Version)

- This version finds a minimum spanning tree from a connected weighted graph.

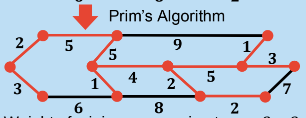
Step 1	Begin creating a tree by selecting a random vertex from the graph.
Step 2	Grow the tree by selecting the closest vertex not yet in the tree. If there is a tie between two or more vertices, pick one at random.
Step 3	Go back to Step 2. Stop when all vertices of the graph are selected such that no cycles are created.

Graph Version Example

(Q1) Find the minimum spanning tree:



Prim's Algorithm



- Weight of minimum spanning tree $= 2+3+5+5+1+4+2+5+2+1+3 =$ **33**

► Topic Is Continued In Next Column ◄

PRIM'S ALGORITHM

Prim's Algorithm (Matrix Version)

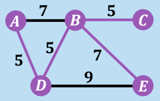
- This version finds a minimum spanning tree from a distance matrix.

Step 1	Select a random vertex, delete its row and mark its column.
Step 2	Scan all marked columns for the lowest non-zero entry and circle that entry. If there is a tie, pick an entry at random.
Step 3	Delete the row containing the circled entry and then mark the new corresponding column.
Step 4	Go back to Step 2. Stop when all rows in the matrix are deleted.

Matrix Version Example

(Q1) Find the minimum spanning tree:

$$\begin{array}{c} \\ A\\B\\C\\D\\E \end{array}\begin{array}{c} \overset{*}{A}\ \ \overset{*}{B}\ \ \overset{*}{C}\ \ \overset{*}{D}\ \ \overset{*}{E} \\ \begin{bmatrix} 0 & 7 & 0 & 5 & 0 \\ 7 & 0 & 5 & \textcircled{5} & 7 \\ 0 & \textcircled{5} & 0 & 0 & 0 \\ \textcircled{5} & 5 & 0 & 0 & 9 \\ 0 & \textcircled{7} & 0 & 9 & 0 \end{bmatrix}\end{array}$$



- Min weight $=$ **22**, Arcs: **AD / DB / BE / BC**

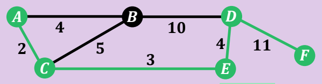
SHORTEST PATH PROBLEM

Finding Shortest Path Between Points

Tip 1	Find all possible paths between each of the two vertices and test each path individually.
Tip 2	Where there are multiple edges, ignore the higher weighted ones.
Tip 3	Sometimes the shortest path doesn't mean the least amount of edges used; check all options.

Shortest Path Problem Example

(Q1) Find the shortest path between A and F.



- Minimum weight $=$ **20**, Path: **ACEDF**

MAXIMUM FLOW & MINIMUM CUT

Flow Network Terminology

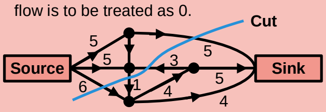
- Source: start point (where flow comes from).
- Sink: end point (where flow finishes).
- Maximum Flow: greatest flow available in a network given the restraints of edge weights.
- Cut: a line drawn through a number of edges which stops all flow from source to sink.
- Minimum Cut: smallest cut of all possible cuts that minimises the sum of edge weights.

Maximum Flow in a Network

Step 1	Find an open path in the network that begins from the source and ends at the sink.
Step 2	Determine the maximum flow that can travel through this open path.
Step 3	Subtract this flow amount from all of the edges in this open path.
Step 4	Go back to Step 1. Stop when all possible open paths are chosen or there is no remaining flow available for another open path.
Step 5	Calculate max flow by adding the flow amount of all paths chosen.

Drawing a Cut in a Network

- Any edge that flows toward the source rather than the sink can be included in a cut and its flow is to be treated as 0.



- Cut Capacity $= 5+0+1+6 =$ **12**

Minimum Cut in a Network

Step 1	Mark all edges that have maximum flow travelling through it.
Step 2	Mark all edges that flow toward the source instead of the sink.
Step 3	Draw a cut that is as close to the source as possible that only passes through edges found in Steps 1 & 2.
Step 4	Calculate the capacity of the cut by adding all of the edges that the cut passes through.

Comparing Max Flow and Min Cut

Maximum Flow = Minimum Cut

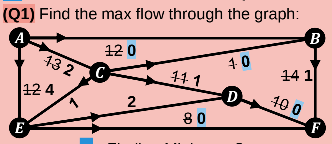
- If the capacity of the minimum cut equals the capacity of the maximum flow, then the calculated maximum flow is verified.

► Topic Is Continued In Next Column ◄

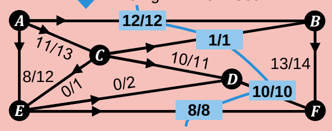
MAXIMUM FLOW & MINIMUM CUT

Max Flow and Min Cut Example

(Q1) Find the max flow through the graph:



Finding Minimum Cut



- Maximum Flow $= 13+10+8 =$ **31**
- Minimum Cut $= 12+1+10+8 =$ **31**
- Max Flow $=$ Min Cut ∴ verified max flow $=$ **31**

CRITICAL PATH ANALYSIS

Critical Path Analysis (*CPA*)

- Minimum Completion Time (*MCT*): least time needed to complete all activities.
- Project Network: a weighted and connected digraph showing all activities for a project.
- Critical Path: sequence of activities that have the longest duration in the project.
- Labeling Activities: **EST | LST**

Earliest Starting Time (*EST*)

- The latest time an activity can be delayed without changing the critical path.
- The earliest time an activity can commence given any predecessors.
- Activity EST's found by forward scanning.

Step 1	Set the EST for any activities with no predecessors as 0.
Step 2	To find EST for the other activities, add the EST from the previous activity to the activity duration. If there are multiple activities feeding into another, choose the highest duration of those activities.
Step 3	Continue forwards through the network (from source to sink).

Latest Starting Time (*LST*)

- The latest time an activity can be delayed without changing the critical path.
- Activity LST's found by backward scanning.

Step 1	Set LST equal to EST of the finishing activity (*a.k.a.* the sink).
Step 2	Using the LST of the sink, work backwards through the network by subtracting the activity duration from the LST of the activity. If there are multiple activities feeding into another, choose the lowest LST of those activities to subtract from.
Step 3	Continue backwards through the network (from sink to source).

Slack / Float of an Activity

Slack for an Activity = LST − EST

- ***Slack*** **:** extra time available that won't change the critical path for a project.
- ***LST*** **:** latest starting time for activity.
- ***EST*** **:** earliest starting time for activity.

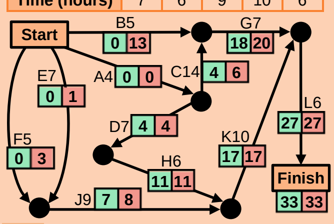
- If slack of an activity is equal to 0, then that activity is on the critical path.

Project Network Example

(Q1a) Use the table to draw a project network.

Activity	A	B	C	D	E	F
Predecessors	-	-	A	A	-	-
Time (hours)	4	5	14	7	7	5

Activity	G	H	J	K	L
Predecessors	B, C	D	E, F	H, J	G, K
Time (hours)	7	6	9	10	6



(Q1b) State the critical path and MCT.

- Critical path $=$ ***D − E − G***, MCT $=$ ***33 hours***

(Q1c) State the EST, LST & slack of activity G.

- EST $=$ **18**, LST $=$ **20**, slack $= 20 - 18 =$ **2**

(Q1d) Out of activities G and J, which one should be chosen to extend for 3 hours in order for the MCT to be kept at a minimum?

- Activity G slack $= 2$, MCT $= 33 + 1 = 34$
- Activity J slack $= 1$, MCT $= 33 + 2 = 35$
- ∴ extend **activity G** to minimise MCT.

HUNGARIAN ALGORITHM

Minimum Cost Assignment

- Assigning tasks to people such that the overall cost is as small as possible.

Step 1	Use the Hungarian Algorithm on the cost matrix.

Maximum Cost Assignment

- Assigning tasks to people such that the overall cost is as large as possible.

Step 1	Subtract every entry in the cost matrix from the largest entry. e.g. $\begin{bmatrix} 1 & 4 & 5 \\ 5 & 7 & 6 \\ 5 & \underline{8} & 3 \end{bmatrix} \to \begin{bmatrix} 7 & 4 & 3 \\ 3 & 1 & 2 \\ 3 & 0 & 5 \end{bmatrix}$
Step 2	Use the Hungarian Algorithm on this new cost matrix.

Hungarian Algorithm

- Can only be used on a square cost matrix.

Step 1	Subtract the smallest entry in each row from all entries in its row.
Step 2	Subtract smallest entry in each column from all entries in its column.
Step 3	Draw straight lines through the rows and columns so that all zero entries are covered. Ensure that minimum number of straight lines are used.
Step 4	If the minimum number of covering lines is equal to the number of rows in the cost matrix, go to Step 6. If the number of covering lines is less than the number of rows in the cost matrix, go to Step 5.
Step 5	Find the smallest entry in the matrix not covered by any line. Subtract this entry from all uncovered entries and add it to all entries covered by a line twice. Return to Step 3.
Step 6	Select and circle a zero entry in each column of the matrix so that no other circled zero entries are in its row.
Step 7	Match the circled entries with the original cost matrix; this is the solution. Note: it is possible to have more than one solution.

Cost Matrix is not a Square Matrix

- The number of columns must equal the number of rows in a cost matrix.

Step 1	Add a dummy row or column of zeroes to the cost matrix so it becomes a square matrix. e.g. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 2 \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix}$
Step 2	Use the Hungarian Algorithm on this new cost matrix.

- Any matching involving the dummy row or column of zeroes added in these steps is to be ignored in the final answer.

Hungarian Algorithm Example

(Q1) Find the minimum cost assignment of the following cost matrix using the Hungarian Algorithm.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} D\quad E\quad F \\ \begin{bmatrix} 8 & 8 & 6 \\ 2 & 3 & 7 \\ 4 & 9 & 3 \end{bmatrix}\end{array}$$

Step 1, Step 2, Step 3

$$\begin{bmatrix} 8 & 8 & 6 \\ 2 & 3 & 7 \\ 4 & 9 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 5 \\ 1 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 5 \\ 1 & 5 & 0 \end{bmatrix}$$

- Step 4: # of lines $= 2$, $2 < 3$, go to Step 5.

Step 5, Step 3

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 5 \\ 1 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 4 & 0 \end{bmatrix}$$

- Step 4: # of lines $= 3$, $3 = 3$, go to Step 6.

Step 6, Step 7

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 8 & 8 & 6 \\ 2 & 3 & 7 \\ 4 & 9 & 3 \end{bmatrix} \to \text{min cost} = 4 + 3 + 6 = \mathbf{13}$$

(Q2) Find the maximum cost assignment of the following cost matrix that isn't square:

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} C\quad D\quad E \\ \begin{bmatrix} 3 & 5 & 2 \\ 6 & 4 & 1 \end{bmatrix}\end{array}$$

Maximum Cost, Square

$$\begin{bmatrix} 3 & 5 & 2 \\ 6 & 4 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 1, Step 2, Step 3

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

- Step 4: # of lines $= 3$, $3 = 3$, go to Step 6.

Step 6, Step 7

$$\begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 5 & 2 \\ 6 & 4 & 1 \end{bmatrix} \to \text{max cost} = 6 + 5 = \mathbf{11}$$

ATAR Math Applications
Units 3 & 4 Exam Notes

Created by **Anthony Bochrinis**